Nonlinear algebra and matrix completion

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Motivation

Problem
Let $\Omega \subseteq [m] \times [n]$. For a given $\Omega$-partial matrix $X \in \mathbb{C}^\Omega$, the low-rank matrix completion problem is

Minimize $\text{rank}(M)$ subject to $M_{ij} = X_{ij}$ for all $(i, j) \in \Omega$

Example
Let $\Omega = \{(1, 1), (1, 2), (2, 1)\}$ and consider the following $\Omega$-partial matrix

$$X = \begin{pmatrix} 1 & 2 \\ 3 & \cdot \end{pmatrix}.$$

Some applications:
- Collaborative filtering (e.g. the “Netflix problem”)
- Computer vision
- Existence of MLE in Gaussian graphical models (Uhler 2012)
State of the art: nuclear norm minimization

The nuclear norm of a matrix, denoted $\| \cdot \|_*$, is the sum of its singular values.

**Theorem (Candès and Tao 2010)**

Let $M \in \mathbb{R}^{m \times n}$ be a fixed matrix of rank $r$ that is sufficiently “incoherent.” Let $\Omega \subseteq [m] \times [n]$ index a set of $k$ entries of $M$ chosen uniformly at random. Then with “high probability,” $M$ is the unique solution to

$$\text{minimize} \quad \|X\|_*$$

subject to $X_{ij} = M_{ij}$ for all $(i, j) \in \Omega$.

The upshot: the minimum rank completion of a partial matrix can be recovered via semidefinite programming if:

- the known entries are chosen uniformly at random
- the completed matrix is sufficiently “incoherent”

Goal: use algebraic geometry to understand the structure of low-rank matrix completion and develop methods not requiring above assumptions.
The algebraic approach

Some subsets of entries of a rank- \( r \) matrix satisfy nontrivial polynomials.

Example

If the following matrix has rank 1, then the bold entries must satisfy the following polynomial

\[
\begin{pmatrix}
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23} \\
  x_{31} & x_{32} & x_{33}
\end{pmatrix}
\]

\[x_{12}x_{21}x_{33} - x_{13}x_{31}x_{11} = 0\]

Király, Theran, and Tomioka propose using these polynomials to:

- Bound rank of completion of a partial matrix from below
- Solve for missing entries

Question

Which subsets of entries of an \( m \times n \) matrix of rank \( r \) satisfy nontrivial polynomials?
Graphs and partial matrices

Subsets of entries of a matrix can be encoded by graphs:

- non-symmetric matrices $\rightarrow$ bipartite graphs
- symmetric matrices $\rightarrow$ semisimple graphs

| $\text{Mat}_{r}^{m\times n}$ | $m \times n$ matrices of rank $\leq r$ | \[
\begin{pmatrix}
5 & \cdot & \cdot \\
-4 & -2 & \cdot \\
\cdot & 8 & 3
\end{pmatrix}
\] | \begin{tikzpicture}
  \node (1) at (0,0) [circle,fill,inner sep=2pt]{\textcolor{white}{1}};
  \node (2) at (1,0) [circle,fill,inner sep=2pt]{\textcolor{white}{2}};
  \node (3) at (2,0) [circle,fill,inner sep=2pt]{\textcolor{white}{3}};
  \draw (1) -- (2);
  \draw (2) -- (3);
\end{tikzpicture}|

| $\text{Sym}_{r}^{n\times n}$ | $n \times n$ symmetric matrices of rank $\leq r$ | \[
\begin{pmatrix}
7 & 4 & \cdot \\
4 & \cdot & 9 \\
\cdot & 9 & 5
\end{pmatrix}
\] | \begin{tikzpicture}
  \node (1) at (0,0) [circle,fill,inner sep=2pt]{\textcolor{white}{1}};
  \node (2) at (1,0) [circle,fill,inner sep=2pt]{\textcolor{white}{2}};
  \node (3) at (2,0) [circle,fill,inner sep=2pt]{\textcolor{white}{3}};
  \draw (1) -- (2);
  \draw (2) -- (3);
\end{tikzpicture}|

- A **$G$-partial matrix** is a partial matrix whose known entries lie at the positions corresponding to the edges of $G$.
- A **completion** of a $G$-partial matrix $M$ is a matrix whose entries at positions corresponding to edges of $G$ agree with the entries of $M$. 
**Generic completion rank**

**Definition**

Given a (bipartite/semisimple) graph $G$, the *generic completion rank of $G$*, denoted $\text{gcr}(G)$, is the minimum rank of a complex completion of a $G$-partial matrix with generic entries.

<table>
<thead>
<tr>
<th>type</th>
<th>$G$</th>
<th>pattern</th>
<th>$\text{gcr}(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>symm</td>
<td></td>
<td>$\begin{pmatrix} a_{11} &amp; ? \ ? &amp; a_{22} \end{pmatrix}$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>symm</td>
<td></td>
<td>$\begin{pmatrix} a_{11} &amp; a_{12} &amp; ? \ a_{12} &amp; a_{22} &amp; a_{23} \ ? &amp; a_{23} &amp; ? \end{pmatrix}$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1 2 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>non</td>
<td></td>
<td>$\begin{pmatrix} a_{11} &amp; a_{12} &amp; ? \ a_{21} &amp; ? &amp; a_{23} \end{pmatrix}$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>r1 c1 r2 c2 c3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Problem
Gain a combinatorial understanding of generic completion rank - how can one use the combinatorics of $G$ to infer $gcr(G)$?

Proposition (Folklore)
Given a bipartite graph $G$, $gcr(G) \leq 1$ iff $G$ has no cycles.

Proposition (Folklore)
Given a semisimple graph $G$, $gcr(G) \leq 1$ iff $G$ has no even cycles, and every connected component has at most one odd cycle.
A cycle in a directed graph is **alternating** if the edge directions alternate.

**Theorem (B.-, 2016)**

Given a bipartite graph $G$, $\text{gcr}(G) \leq 2$ if and only if there exists an acyclic orientation of $G$ that has no alternating cycle.
A cycle in a directed graph is \textit{alternating} if the edge directions alternate.

\textbf{Theorem (B.-., 2016)}

Given a bipartite graph $G$, $\text{gcr}(G) \leq 2$ if and only if there exists an acyclic orientation of $G$ that has no alternating cycle.
Proof sketch

Theorem (B.-, 2016)

Given a bipartite graph \( G \), \( gcr(G) \leq 2 \) if and only if there exists an acyclic orientation of \( G \) that has no alternating cycle.

- Rephrase the question: describe the independent sets in the algebraic matroid underlying the variety of \( m \times n \) matrices of rank at most 2
- This algebraic matroid is a restriction of the algebraic matroid underlying a Grassmannian \( Gr(2, N) \) of affine planes
- Algebraic matroid structure is preserved under tropicalization
- Apply Speyer and Sturmfels’ result characterizing the tropicalization of \( Gr(2, N) \) in terms of tree metrics to reduce to an easier combinatorial problem

Open question

Does there exist a polynomial time algorithm to check the combinatorial condition in the above theorem, or is this decision problem NP-hard?
Issue: real vs complex

What happens when you only want to consider real completions?

Definition

Given a bipartite or semisimple graph $G$, there may exist multiple open sets $U_1, \ldots, U_k$ in the space of real $G$-partial matrices such that the minimum rank of a completion of a partial matrix in $U_i$ is $r_i$. We call the $r_i$s the typical ranks of $G$.

The graph $\bullet \ \bullet$ has typical ranks 1 and 2.

\[
\begin{pmatrix}
a_{11} & \cdot \\
\cdot & a_{22}
\end{pmatrix}
\]

In a completion to rank 1, the missing entry $t$ must satisfy $a_{11}a_{22} - t^2 = 0$. 
Facts about typical ranks

Proposition (B.-Blekherman-Sinn 2018)

Let $G$ be a bipartite or semisimple graph.

1. The minimum typical rank of $G$ is $\text{gcr}(G)$.
2. The maximum typical rank of $G$ is at most $2\text{gcr}(G)$.
3. All integers between $\text{gcr}(G)$ and the maximum typical rank of $G$ are also typical ranks of $G$.

See also Bernardi, Blekherman, and Ottaviani 2015 and Blekherman and Teitler 2015.
Case study: disjoint union of cliques

Let $K_m \sqcup K_n$ denote the disjoint union of two cliques with all loops.

$$K_3 \sqcup K_4 = \begin{array}{c}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\end{array}$$

$$\begin{pmatrix}
a_{11} & a_{12} & a_{13} & ? & ? & ? & ? \\
a_{12} & a_{22} & a_{23} & ? & ? & ? & ? \\
a_{13} & a_{23} & a_{33} & ? & ? & ? & ? \\
? & ? & ? & a_{44} & a_{45} & a_{46} & a_{47} \\
? & ? & ? & a_{45} & a_{55} & a_{56} & a_{57} \\
? & ? & ? & a_{46} & a_{56} & a_{66} & a_{67} \\
? & ? & ? & a_{47} & a_{57} & a_{67} & a_{77}
\end{pmatrix}$$

Proposition (B.-Blekherman-Lee)

The generic completion rank of $K_m \sqcup K_n$ is $\max\{m, n\}$. The maximum typical rank of $K_m \sqcup K_n$ is $m + n$. 
Proposition (B.-Blekherman-Lee)

The generic completion rank of $K_m \sqcup K_n$ is $\max\{m, n\}$. The maximum typical rank of $K_m \sqcup K_n$ is $m + n$.

A $(K_m \sqcup K_n)$-partial matrix looks like:

$$M = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}.$$

By Schur complements:

$$\text{rank}(M) = \text{rank}(A) + \text{rank}(B - X^T A^{-1} X).$$

If $A \prec 0$ and $B \succ 0$, then $\det(B - X^T A^{-1} X) > 0$ for real $X$.

Corollary

Every integer between $\max\{m, n\}$ and $m + n$ is a typical rank of $K_m \sqcup K_n$. 
Case study: disjoint union of cliques

Given real symmetric matrices $A$ and $B$ of full rank, of possibly different sizes:

- $p_A$ ($p_B$) denotes the number of positive eigenvalues of $A$ ($B$)
- $n_A$ ($n_B$) denotes the number of negative eigenvalues of $A$ ($B$)
- the **eigenvalue sign disagreement of $A$ and $B$** is defined as:

$$\text{esd}(A, B) := \begin{cases} 
0 & \text{if } (p_A - p_B)(n_A - n_B) \geq 0 \\
\min\{|p_A - p_B|, |n_A - n_B|\} & \text{otherwise}
\end{cases}$$

**Theorem (B.-Blekherman-Lee)**

Let $M = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}$ be a generic real $K_m \sqcup K_n$-partial matrix. Then $M$ is minimally completable to rank $\max\{m, n\} + \text{esd}(A, B)$. 
When full rank is typical

**Theorem (B.-Blekherman-Lee)**

Let $G$ be a semisimple graph on $n$ vertices. Then $n$ is a typical rank of $G$ if and only if the complement graph of $G$ is bipartite.

If the complement is bipartite, then $n$ is a typical rank:

$$M = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}$$

By Schur complements:

$$\text{rank}(M) = \text{rank}(A) + \text{rank}(B - X^T A^{-1} X),$$

so if $A \prec 0$ and $B \succ 0$, then $\det(B - X^T A^{-1} X)$ is strictly positive.
When full rank is typical

**Theorem (B.-Blekherman-Lee)**

Let $G$ be a semisimple graph on $n$ vertices. Then $n$ is a typical rank of $G$ if and only if the complement graph of $G$ is bipartite.

If complement is *not* bipartite, then $n$ is *not* a typical rank:

- A graph is bipartite if and only if it is free of odd cycles
- If complement graph *is* an odd cycle, then determinant of a $G$-partial matrix, viewed as a polynomial in the unknown entries, has odd degree
- Deleting edges from a graph will not increase maximum typical rank.

\[
\begin{pmatrix}
  a_{11} & x & a_{13} & a_{14} & t \\
x & a_{22} & y & a_{24} & a_{25} \\
a_{13} & y & a_{33} & z & a_{35} \\
a_{14} & a_{24} & z & a_{44} & w \\
t & a_{25} & a_{35} & w & a_{55}
\end{pmatrix}
\]
Typical ranks for nonsymmetric matrices: some examples

The following bipartite graph has 2 and 3 as typical ranks.

\[
\begin{pmatrix}
? & a_{12} & a_{13} & a_{14} \\
 a_{21} & ? & a_{23} & a_{24} \\
a_{31} & a_{32} & ? & a_{34} \\
a_{41} & a_{42} & a_{43} & ?
\end{pmatrix}
\]

Let \( \text{mtr}(G) \) denote the maximum typical rank of \( G \).

**Theorem (B.-Blekherman-Sinn)**

Let \( G \) be obtained by gluing two bipartite graphs \( G_1 \) and \( G_2 \) along a complete bipartite subgraph \( K_{m,n} \). If

\[
\max\{ \text{mtr}(G_1), \text{mtr}(G_2) \} \geq \max\{ m, n \},
\]

then \( \text{mtr}(G) = \max\{ \text{mtr}(G_1), \text{mtr}(G_2) \} \). The same is true for generic completion rank.

**Open question**

Does there exist a bipartite graph that has more than two typical ranks?
Empty $k$-cores

The $k$–core of a graph $G$ is the graph obtained by iteratively removing vertices of degree $k - 1$ or less. The 2-core of the graph below is empty.

\[
\begin{align*}
\text{\textbullet} & \quad \rightarrow \quad \text{\textbullet} \\
\text{\textbullet} & \quad \rightarrow \quad \text{\textbullet} \\
\text{\textbullet} & \quad \rightarrow \quad \text{\textbullet} \\
\end{align*}
\]

Theorem (B.-, Blekherman, Sinn)

Let $G$ be bipartite. If the $k$-core of $G$ is empty, then all typical ranks of $G$ are at most $k - 1$.

Corollary

Let $G$ be bipartite. Then the maximum typical rank of $G$ is $2 \text{gcr}(G) - 1$.

Open question

Which bipartite graphs of generic completion rank 2 also have 3 as a typical rank?
Conclusion

- All generic $G$-partial matrices can be completed to rank $\text{gcr}(G)$ over $\mathbb{C}$
- We can characterize all the bipartite graphs with generic completion rank $\leq 2$ (semisimple case is still open)
- Over the reals, a graph can have many typical ranks

Open problems:
- Find a polynomial-time algorithm to decide if a given bipartite graph has an acyclic orientation with no alternating cycle, or prove that this decision problem is NP-hard
- Find a bipartite graph that exhibits three or more typical ranks
- Characterize the graphs with generic completion rank 2 that also exhibit 3 as a typical rank
A. Bernardi, G. Blekherman, and G. Ottaviani.
On real typical ranks.

**Daniel Irving Bernstein.**
Completion of tree metrics and rank-2 matrices.
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On maximum, typical and generic ranks.

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Franz Király, Louis Theran, and Ryota Tomioka.
The algebraic combinatorial approach for low-rank matrix completion.